

Collective modes in multiband superfluids and superconductors: Multiple dynamical classes

Yukihiro Ota,^{1,2} Masahiko Machida,^{1,2,3} Tomio Koyama,^{2,4} and Hideo Aoki^{3,5}

¹ CCSE, Japan Atomic Energy Agency, Higashi-Ueno, Tokyo 110-0015, Japan

² CREST(JST), Honcho, Kawaguchi, Saitama 332-0012, Japan

³ JST, TRIP, Sanbancho, Tokyo 102-0075, Japan

⁴ Institute for Materials Research, Tohoku University, Katahira, Sendai 980-8577, Japan

⁵ Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan

(Dated: February 25, 2011)

One important way to characterize the states having a gauge symmetry spontaneously broken over multibands should be to look at their collective excitation modes. We find that a three-band system has *multiple Leggett modes* with significantly different masses, which can be classified into different dynamical classes according to whether multiple inter-band Josephson currents add or cancel. This provides a way to dynamically characterize multiband superconductivity while the pairing symmetry is a static property.

PACS numbers: 74.20.-z, 03.75.Kk

Introduction — Superconductivity and superfluidity, with their spontaneous broken gauge symmetry, harbor unexpected fascinations. Specifically, the seminal discovery of the iron-based superconductor¹ has kicked off renewed interests in multiband superconductivity. There, the gauge-symmetry breaking involves multiple bands, so the ground state is expected to possess the features that can occur only in multiband systems, such as the s_{\pm} wave pairing proposed for the iron-based superconductor². Even more interesting are excited states, especially the collective excitations of condensate phases associated with the broken gauge symmetry³. For two-band superconductors, there is a classical 1966 work by Leggett, who has shown that a two-band superconductor accommodates a special collective excitation, now called Leggett's mode, emerging as an out-of-phase mode between different superfluids⁴. The Leggett's mode was experimentally detected in a two-band superconductor, MgB₂ (Ref.⁵) via the Raman scattering⁶.

However, little is known as to what happens to the collective modes when there are three or more bands. In fact, despite intensive studies on the gap function symmetries in the ground states for various classes of superconductors, dynamics of collective excitations in multiband systems has yet to be systematically investigated. The question is becoming increasingly realistic, since the iron-based superconductors, for instance, have been revealed to have a five-band structure, where three bands contribute to both the Fermi surface and the gap function², as subsequently experimentally examined⁷. To study the dynamics of collective modes in multiband systems is thus expected to be a new avenue together with unconventional pairing symmetry.

The purpose of the present paper is to examine, in general, whether a novel class of Leggett's modes indeed exists specifically in superconductors and superfluids that have three or more bands, which should shed light on dynamics of multiband condensates. Naively, one might expect that the two- and three-band cases may be simi-

lar. Here, however, we shall show, on the basis of an effective action for the phase fluctuations, that multiband superfluids having three or more bands are in fact unique in that there exist multiple Leggett's modes classified by a dynamical class introduced here. The distinction between the dynamical classes come from the presence of multiple inter-band Josephson couplings, whose additive and subtractive combination to the effective action are, respectively, classified as class "even" and "odd." The two Leggett's modes in the class odd are predicted to have significantly different masses, which is testable and serves to characterize multiband superconductors.

Formulation — Let us start with a model for multiband superfluidity/superconductivity. We take the simplest possible Bardeen-Cooper-Schrieffer type Hamiltonian density for an N -band superfluid, assumed to be neutral here, is given as $\hat{\mathcal{H}}(\mathbf{r}) = \hat{\mathcal{H}}_{\text{kinetic}}(\mathbf{r}) + \hat{\mathcal{H}}_{\text{pairing}}(\mathbf{r})$ with $\hat{\mathcal{H}}_{\text{kinetic}} = \sum_{i=1}^N \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}_{i\sigma}^{\dagger} \varepsilon_i (-i\nabla) \hat{\psi}_{i\sigma}$ where the dispersion $\varepsilon_i(\mathbf{k})$ is assumed to be parabolic with $\hbar = 1$, while $\hat{\mathcal{H}}_{\text{pairing}} = -\sum_{i,j=1}^N g_{ij} \hat{\psi}_{i\uparrow}^{\dagger} \hat{\psi}_{i\downarrow}^{\dagger} \hat{\psi}_{j\downarrow} \hat{\psi}_{j\uparrow}$. Here $\hat{\psi}_{i\sigma}$ ($i = 1, 2, \dots, N$) is the field operator for the i th band in a superconductor or i th atomic species in an atomic gas, and $\mathcal{G} \equiv (g_{ij})$ the pairing matrix, taken to be positive-definite, and we drop its \mathbf{k} -dependence. While \mathcal{G} may come from (the short-range part of) the electron correlation, we ignore the Coulombic part of the interaction, since the mass of Leggett's modes is not affected by the Anderson-Higgs mechanism, although the group velocity of the mode reflects the Coulomb interaction in superconductors as shown in Ref.⁸ for $N = 2$. While $\hat{\mathcal{H}}_{\text{pairing}}$ with \mathbf{k} -dependence dropped, as is done in Leggett's original treatment, is too simple to examine complicated band structures as in the iron-based superconductors, we should initially know the basic physics for collective excitations in multiband superfluidity/superconductivity in the simplest case.

The grand partition function is given, with the imaginary-time functional integral method,

in terms of a set of auxiliary fields $\Delta^{(i)}$ as $Z = Z_0 \int \prod_{i=1}^N \mathcal{D}\Delta^{(i)} \mathcal{D}\Delta^{(i)*} e^{-S_{\text{eff}}}$. Here the effective action is $S_{\text{eff}} = \int_0^\beta d\tau \int d^3\mathbf{r} \sum_{i,j=1}^N \Delta^{(i)*} (\mathcal{G}^{-1})_{ij} \Delta^{(j)} - \sum_{i=1}^N \text{Tr} \ln \hat{G}_{0,i} - \sum_{i=1}^N \text{Tr} \ln \hat{G}_i^{-1}$, where we assume the inverse \mathcal{G}^{-1} is well-defined (i.e., $\det \mathcal{G} > 0$), Z_0 the partition function for the non-interacting system, and β the inverse temperature. \hat{G}_i is Green's function in the Nambu representation for the i th-band, which satisfies $\hat{G}_i^{-1} = \hat{G}_{0,i}^{-1}(\hat{I} - \hat{G}_{0,i}\hat{K}_i)$, where $\hat{G}_{0,i}$ is the free-fermion Green's function, \hat{I} the unit matrix, and $\hat{K}_i = \sigma_1 \text{Re}\Delta^{(i)} - i\sigma_2 \text{Im}\Delta^{(i)}$, with σ_α being the Pauli matrices. The gaps are coupled through the inverse of the pairing matrix (\mathcal{G}^{-1}) , and we shall see that the non-zero off-diagonal elements of \mathcal{G}^{-1} determine the Leggett's modes.

With this effective action the static gap equation reads

$$\Delta_i = \sum_{j=1}^N g_{ij} \Delta_j N_j \int_0^{\omega_c} d\xi \frac{\tanh(\beta E_j/2)}{E_j}, \quad (1)$$

where $E_i \equiv \sqrt{\xi^2 + |\Delta_i|^2}$, ω_c a cut-off frequency, and N_j the density of states (DOS) of the j th fermion on the Fermi surface. Here, we assume that each of the $\Delta^{(i)}$'s is constant (an s-wave). We can then look at the phase $\varphi_0^{(i)}$ in $\Delta_i = |\Delta_i| e^{i\varphi_0^{(i)}}$. The gap functions on different bands can take either the same or opposite signs, i.e., the phase difference, $\phi_0^{(i,j)} \equiv \varphi_0^{(j)} - \varphi_0^{(i)}$, takes either 0 or $\pm\pi$ and should obviously satisfy $\sum_{i=1}^N \phi_0^{(i,i+1)} \equiv 0 \text{ mod } 2\pi$ with $\varphi_0^{(N+1)} \equiv \varphi_0^{(1)}$, as depicted in Fig.1(a).

Let us derive an effective action for the fluctuations in the superfluid phase to single out the collective dynamics at zero temperature. We first decompose the phase $\varphi_0^{(i)} + \varphi^{(i)}$, into the equilibrium $\varphi_0^{(i)}$ (as obtained in the mean-field gap equation) and the phase fluctuation $\varphi^{(i)}$. Around the solution of Eq. (1), we obtain an action, $S_{\text{eff}} = \int_0^\beta d\tau \int d\mathbf{r} V_{\text{interband}} + \sum_{i=1}^N \sum_{m=1}^\infty \text{Tr} (\hat{G}'_{0,i} \hat{K}'_i)^m / m$, where

$$V_{\text{interband}} = \frac{1}{2} \sum_{i < j} \eta_{ij} \lambda_{ij} [1 - \cos(\varphi^{(j)} - \varphi^{(i)})], \quad (2)$$

$$\eta_{ij} = \cos(\phi_0^{(i,j)} + \kappa_{ij}), \quad \lambda_{ij} = 4|(\mathcal{G}^{-1})_{ij}| |\Delta_i| |\Delta_j|, \quad (3)$$

comprises a sum of Josephson-couplings between the phases of different superfluid gaps that represents the inter-band Josephson currents caused by the relative phase fluctuations. We mention that the inter-band Josephson couplings are analogous quantiles with the standard Josephson couplings but derived in non-perturbative manner. In the above expression, the constraint $\phi_0^{(i,j)} = 0$ or $\pm\pi$ is used, and $\kappa_{ij} (= 0 \text{ or } \pi)$ the sign of $(\mathcal{G}^{-1})_{ij}$, i.e., $e^{i\kappa_{ij}} \equiv -\text{sgn}(\mathcal{G}^{-1})_{ij}$. The primed quantities are gauge-transformed, i.e., $\hat{G}'_{0,i}(x; x') \equiv \hat{U}_i(x) \hat{G}_{0,i}(x; x') \hat{U}_i^\dagger(x')$, where $\hat{U}_i = e^{-i\sigma_3(\varphi_0^{(i)} + \varphi^{(i)})/2}$ and

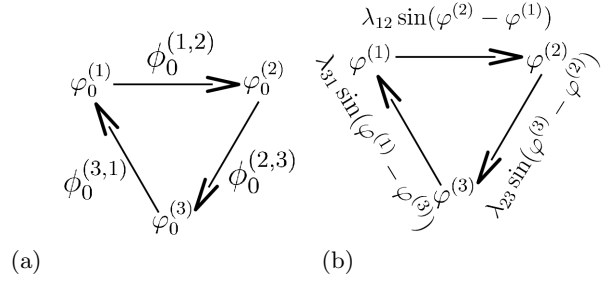


FIG. 1: Schematic diagrams for $N = 3$ for (a) the differences in the superfluid phases in equilibrium $\varphi_0^{(i)}$ and (b) the inter-band Josephson coupling for the phase fluctuation $\varphi^{(i)}$ around the equilibrium (where arrows represent the phase difference associated with the inter-band Josephson current).

class	$(\eta_{12}, \eta_{23}, \eta_{31})$	parity of $p_{\vec{\kappa}}$	parity of $p_{\vec{\phi}_0}$
even	(1, 1, 1)	even	even
odd	(1, 1, -1)	odd	even
odd	(1, -1, 1)	odd	even
odd	(-1, 1, 1)	odd	even

TABLE I: Classification of the signs of the coupling for $N = 3$.

$x \equiv (\tau, \mathbf{r})$, with the Dyson equation transformed into $\hat{G}'_i = \hat{G}'_{0,i}(\hat{I} - \hat{G}'_{0,i}\hat{K}'_i)$.

Parity in the inter-band Josephson couplings — Let us begin with a trivial two-band ($N = 2$) case, which is relevant to the superconductivity in MgB_2 ⁵. Equation (2) then reduces to $V_{\text{interband}} = (\eta_{12}\lambda_{12}/2)[1 - \cos(\varphi^{(2)} - \varphi^{(1)})]$ with a single inter-band Josephson coupling. The sign of $-(\mathcal{G}^{-1})_{12} (= g_{12}/\det \mathcal{G})$ is equal to that of g_{12} . Hence $\kappa_{12} = 0$ when $g_{12} > 0$, for which $\phi_0^{(1,2)} = 0$ for $V_{\text{interband}}$ to give a stable gap solution according to Eq. (3). Similarly, $\kappa_{12} = \phi_0^{(1,2)} = \pi$ when $g_{12} < 0$. Thus we always have $\eta_{12} = 1$, which implies that the sign of g_{12} is totally irrelevant to the spectrum of collective modes in the two-band case, as noted in Refs. 4,8,9.

We now turn to the three-band case with $N = 3$. We have three kinds of the inter-band Josephson currents as schematically depicted in Fig. 1(b). In this case the set of the signs of the couplings $(\eta_{12}, \eta_{23}, \eta_{31})$ can be classified into: (i) all the signs positive, (ii) two positive, one negative, (iii) one positive, two negative, and (iv) all negative. We note that, when two or three η_{ij} 's are negative, the Hessian matrices associated with $V_{\text{interband}}$ always have negative eigenvalues at $(\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}) = (0, 0, 0)$ ¹⁰, which implies that the solution of Eq. (1) is not a stable minimum of S_{eff} . We can thus exclude cases (iii) and (iv) [Table I].

Let us have a closer look at $\{\phi_0^{(i,j)}\}$ and $\{\kappa_{ij}\}$, from which $\{\eta_{ij}\}$ in Eq. (3) is determined. Note first that $\{\kappa_{ij}\}$ is governed by \mathcal{G}^{-1} rather than \mathcal{G} , where, e.g., $-(\mathcal{G}^{-1})_{12} = (g_{12}g_{33} - g_{13}g_{23})/\det \mathcal{G}$ for $N = 3$. If we introduce a shorthand, $\vec{\kappa} = (\kappa_{12}, \kappa_{23}, \kappa_{31})$, we can define

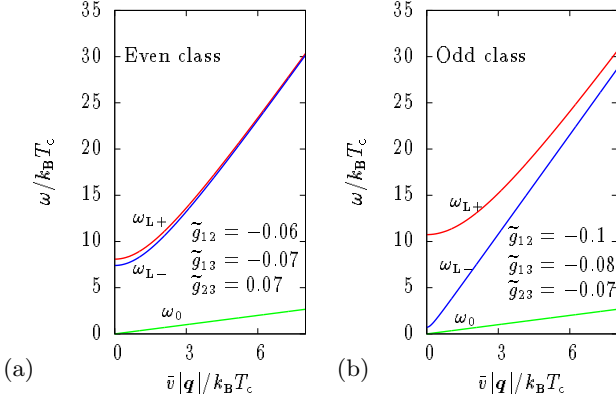


FIG. 2: (Color online) Dispersion relations for ω_{L+} (red line), ω_{L-} (blue) and ω_0 (green) for (a) the class even with the values of \tilde{g}_{ij} displayed for which $\Delta_2/\Delta_1 = -1.00$ and $\Delta_3/\Delta_1 = -1.01$, and (b) the class odd with $\Delta_2/\Delta_1 = -0.73$, $\Delta_3/\Delta_1 = -0.38$. We set $\omega_c/k_B T_c = 5000$ and $\tilde{g}_{ii} = \frac{1}{3}$. The DOS (N_i) and the Fermi velocity are assumed to be identical between the bands.

an integer,

$$p_{\vec{\kappa}} \equiv \kappa_{12} + \kappa_{23} + \kappa_{31} \pmod{2\pi}. \quad (4)$$

Since $\kappa_{ij} = 0$ or π , we have either $p_{\vec{\kappa}}/\pi = 0$ or 1 , which defines the parity “even” and “odd” classes, respectively. For the phases $\vec{\phi}_0 = (\phi_0^{(1,2)}, \phi_0^{(2,3)}, \phi_0^{(3,1)})$, we have four possible cases, $\vec{\phi}_0 = (0, 0, 0)$, $(0, \pi, \pi)$, $(0, \pi, -\pi)$, or $(0, -\pi, \pi)$ as seen in Fig. 1(a), where we assume $\phi_0^{(1,2)} = 0$ without a loss of generality. The first case corresponds to an s -wave with no sign changes, while the others sign-reversing s -waves. If we define $p_{\vec{\phi}_0} \equiv \phi_0^{(1,2)} + \phi_0^{(2,3)} + \phi_0^{(3,1)} \pmod{2\pi}$, we have always $p_{\vec{\phi}_0} \equiv 0$.

In the class even, in which all η_{ij} ’s are 1, we find, for all the above possibilities for $\vec{\kappa}$ and $\vec{\phi}_0$, that the class even occurs only when $p_{\vec{\kappa}} \equiv 0$. On the other hand, the class odd, where one of η_{ij} ’s is -1 , should satisfy $p_{\vec{\kappa}} \equiv 1$. Thus the class for $\{\eta_{ij}\}$ are completely characterized by the parity of $p_{\vec{\kappa}}$ alone, as summarized in Table I. We remark that $\text{Tr}(\hat{G}'_{i,0} \hat{K}'_i)^m$ appearing in S_{eff} does *not* depend on $\{\eta_{ij}\}$, which implies that the effective action is completely distinguished by the dynamical class characterized by the parity of $p_{\vec{\kappa}}$. Such dynamical classification should always be applicable in the pairing-interaction parameter space when $N \geq 3$. In other words, $\{\kappa_{ij}\}$, which does not have to coincide with $\{\phi_0^{(i,j)}\}$ modulo 2π , is *no longer* a matter of convention unlike the case of $N = 2$.

Collective modes — We are now in position to calculate the collective modes. If we expand S_{eff} around the stable solution of Eq. (1) for $\varphi^{(i)}(q)$ with $q \equiv (i\nu_\ell, \mathbf{q})$ and $\nu_\ell = 2\ell\pi/\beta$ ($\ell \in \mathbb{Z}$) the Matsubara frequency, we have, to the leading order, $S_{\text{eff}} \approx \sum_q \tilde{\varphi}_q (\mathcal{M}_q/4) \tilde{\varphi}_q$ in the long-wavelength limit ($|\mathbf{q}| \rightarrow 0$)⁸. Here we have introduced $\tilde{\varphi}_q = {}^t(\varphi^{(1)}(q), \varphi^{(2)}(q), \varphi^{(3)}(q))$, and a 3×3 real symmet-

ric matrix,

$$\mathcal{M}_q = \begin{pmatrix} M_{11}(q) + \mu_{11} & -\eta_{12}\lambda_{12} & -\eta_{31}\lambda_{31} \\ -\eta_{12}\lambda_{12} & M_{22}(q) + \mu_{22} & -\eta_{23}\lambda_{23} \\ -\eta_{31}\lambda_{31} & -\eta_{23}\lambda_{23} & M_{33}(q) + \mu_{33} \end{pmatrix},$$

where $M_{ii}(q) = N_i \nu_\ell^2 + \sum_\xi (N_i v_{F\xi,i}^2/3) q_\xi^2$ with $v_{F\xi,i}$ the ξ th component ($\xi = x, y, z$) of the i th fermion’s Fermi velocity. The diagonal elements involve the contributions from the inter-band Josephson currents, $\mu_{11} = \eta_{12}\lambda_{12} + \eta_{31}\lambda_{31}$, etc. The dispersion relations for the collective excitation modes are given by the roots of $\det \mathcal{M}_q = 0$. After an analytic continuation $i\nu_\ell \rightarrow \omega + i0$, we obtain the three roots, ω_0 , ω_{L+} , and ω_{L-} . Figure 2 displays the full dispersion. We can give explicit formulae for the three collective modes to the leading order in q for isotropic systems. The first root ω_0 corresponds to the Nambu-Goldstone mode with $\omega_0 = V|\mathbf{q}| + O(|\mathbf{q}|^2)$, where $V^2 = \sum_{i=1}^3 (N_i/N_{\text{tot}})(v_{Fi}^2/3)$ and $N_{\text{tot}} = \sum_i N_i$. The other two roots, ω_{L+} and ω_{L-} , are the Leggett’s modes $\omega_{L\pm}^2 = \nu_\pm^2 + c_\pm^2 |\mathbf{q}|^2 + O(|\mathbf{q}|^4)$, where $c_\pm^2 = \pm U^2 \mp \nu_\mp^2 V'^2 / (\nu_+^2 - \nu_-^2)$. Here the “mass gap” (the Leggett’s mode frequency at $|\mathbf{q}| = 0$, i.e., ν_+, ν_-) are characterized by two quantities: The frequency scale is given by $\bar{\nu}^2 = \eta_{12}\nu_{12}^2 + \eta_{23}\nu_{23}^2 + \eta_{31}\nu_{31}^2$ and $\nu_{ij}^2 = (N_i + N_j)\lambda_{ij}/N_i N_j$, where ν_{ij} for $\lambda_{ij} \neq 0$ corresponds to the frequency of the Leggett’s mode in the two-band model. The difference $\nu_+ - \nu_- (\propto \sqrt{1-D})$ is characterized by

$$D = \frac{4N_{\text{tot}}(\eta_{12}\eta_{31}\lambda_{12}\lambda_{31} + \eta_{12}\eta_{23}\lambda_{12}\lambda_{23} + \eta_{23}\eta_{31}\lambda_{23}\lambda_{31})}{\bar{\nu}^4 N_1 N_2 N_3}.$$

For $D \rightarrow 1$ the mass difference $\nu_+ - \nu_-$ vanishes, whereas for $D \rightarrow 0$ ω_{L-} becomes a gapless mode (i.e., $\omega_{L-} \rightarrow c_- |\mathbf{q}|$). We find that $\nu_+ - \nu_-$ becomes large due to a cancellation among the terms in the numerator of D occurs for class odd. The term quadratic in $|\mathbf{q}|$ for the two Leggett’s modes are characterized by c_\pm , which involve two velocities, $U^2 = (1-D)^{-1/2} \sum_{i=1}^3 (\mu_{ii}/N_i \bar{\nu}^2)(v_{Fi}^2/3)$ and $V'^2 = \bar{\nu}^2/3 - V^2$, with $\bar{\nu}^2 = \sum_{i=1}^3 v_{Fi}^2$. In the limit where two bands are decoupled ($\lambda_{23} = \lambda_{31} = 0$), we recover the two-band result⁸ with two gapless modes (ω_0, ω_{L-}), and one gapped mode (ω_{L+}), along with ν_+ and ν_- that coincide with those in Ref.¹¹.

Having formulated the Leggett’s modes, let us see how they reflect the difference in the dynamical classes even and odd. If we look at the simplest case of $\tilde{N}_i \equiv N_i/N_{\text{tot}} = \frac{1}{3}$, and $\tilde{v}_{Fi}^2 \equiv v_{Fi}^2/\bar{\nu}^2 = \frac{1}{3}$ (for $i = 1, 2, 3$), for $\tilde{g}_{11} = \tilde{g}_{22} = \tilde{g}_{33} = \frac{1}{3}$ with $\tilde{g}_{ij} \equiv g_{ij}/\text{Tr} \mathcal{G}$, Fig. 2 displays typical dispersion relations for ω_0 and $\omega_{L\pm}$ for the two dynamical classes. We immediately notice that the mass difference $\nu_+ - \nu_- (= \omega_{L+} - \omega_{L-}$ at $|\mathbf{q}| = 0$) is much greater in the class odd than in the class even. Systematic variation of $\nu_+ - \nu_-$ on an interaction parameter space ($\tilde{g}_{12}, \tilde{g}_{13}$) is shown, first in Fig. 3(a) for the case of two negative (repulsive) and one positive (attractive) inter-band couplings. The allowed parameter region is restricted by the positive-definiteness of \mathcal{G} ¹². The mass

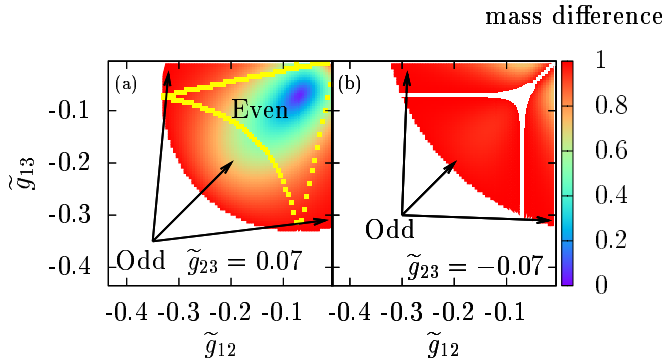


FIG. 3: (Color online) Color-coded mass difference ($\propto \sqrt{1-D}$) in the multiple Leggett's modes plotted on \tilde{g}_{12} - \tilde{g}_{31} plane for $\tilde{g}_{ii} = \frac{1}{3}$. We assume identical normal state properties between the three bands. (a) Case of two negative (repulsive) and one positive (attractive) inter-band couplings. Yellow lines represent the boundaries between the even and odd classes. (b) Case of all the three inter-band couplings negative (repulsive), where only the class odd is allowed. In the blank area, the Leggett's modes become unstable.

difference is close to zero in the central area of the class even region, while it is relatively large in the class odd region. Thus, we recognize that the mass difference of the Leggett's modes is an indicator of the difference in the dynamical classes. Note, however, that the distinction becomes blurred when the three-band system approaches a two-band behavior, which occurs when one of the off-diagonal elements of \mathcal{G}^{-1} is much greater than the others (e.g., around the corners of the class odd region with a triangular shape), so that there is no jumps at the boundary.

Next, Fig. 3(b) displays the result for the case of all the three inter-band couplings negative (repulsive) with $g_{12}, g_{13}, g_{23} < 0$. Notably, this case always has the class odd throughout. This implies that the mass difference of the Leggett's modes is *lower bounded*. The result shows that the bound is $\simeq 0.7$ for $\tilde{g}_{23} = -0.07$ and $g_{ii} = \frac{1}{3}$. Even more interesting, we find that the Leggett's modes

become unstable or ill-defined (within the present treatment which assumes real Δ_i 's) in a narrow but finite region (blank area in Fig. 3(b)) where all three (or two) of g_{12}, g_{13}, g_{23} take similar values. In this case, every two among the three gap functions $\Delta_1, \Delta_2, \Delta_3$ in Eq. (1) want to have opposite signs, but end up with complex values (i.e., the relative phase differences between the gaps deviate from 0 or π) due to the “frustration”, as pointed out in Ref. ¹³.

Summary — We have shown the presence of multiple dynamical classes in the $N(\geq 3)$ -band superfluidity, which are characterized in terms of the parity of the multiple inter-band Josephson couplings. We have revealed that the mass difference of the Leggett's modes is greater in the class odd, in contrast to the two-band case where the classification does not exist. So, the behavior of the multiple Leggett's modes is expected to characterize the dynamics of excitations in multiband superconductors and superfluids. The expressions for the parities for $N = 3$ given here can be extended to $N \geq 4$, with their classification being an interesting future problem. Other future works include an extension of the present discussion to more general, \mathbf{k} -dependent case for realistic description of multiband superconductors, such as the iron-based superconductors with their material dependence², and an extension to anisotropic pairings. Another intriguing problem is the Leggett's modes in the case where the gap functions break the time-reversal symmetry in “frustrated” three-band superconductivity¹³. Multiband superfluidity in cold fermionic atomic gases with multiple hyperfine states may also be an interesting playing ground where we have a greater tunability due to the Feshbach resonance⁹. How to detect Leggett's mode, for which Raman spectra have been discussed in Ref. ^{14–17}, is also an important problem.

YO and MM wish to thank illuminating discussions with Y. Yamanaka, H. Nakamura, M. Okumura, N. Nakai, Y. Nagai, and M. Mine. HA acknowledges valuable discussions with A. Leggett, K. Kuroki and R. Arita.

¹ Y. Kamihara, T. Watanabe, M. Hirano, and H. Hosono, J. Am. Chem. Soc. **130**, 3296 (2008).

² K. Kuroki *et al.*, Phys. Rev. Lett. **101**, 087004 (2008); **102**, 109902(E) (2009); K. Kuroki *et al.*, Phys. Rev. B **79**, 224511 (2009).

³ H. Umezawa, *Advanced Field Theory: Micro, Macro, and Thermal Physics* (AIP press, New York, 1993).

⁴ A. J. Leggett, Prog. Theor. Phys. **36**, 901 (1966).

⁵ X. X. Xi, Rep. Prog. Phys. **71**, 116501 (2008).

⁶ G. Blumberg *et al.*, Phys. Rev. Lett. **99**, 227002 (2007).

⁷ K. Ishida, Y. Nakai, and H. Hosono, J. Phys. Soc. Jpn. **78**, 062001 (2009).

⁸ S. G. Sharapov, V. P. Gusynin, and H. Beck, Eur. Phys. J. B **30**, 45 (2002).

⁹ M. Iskin and C. A. R. Sá de Melo, Phys. Rev. B **74**, 144517

(2006).

¹⁰ If a hermitian matrix has a negative and non-zero diagonal element, it has a negative and non-zero eigenvalue.

¹¹ F. G. Korchorbe and M. E. Palistrant, Physica C **298**, 217 (1998).

¹² G. Kimura, Phys. Lett. A **314**, 339 (2003).

¹³ V. Stanev and Z. Tešanović, Phys. Rev. B **81**, 134522 (2010).

¹⁴ A. V. Chubukov, I. Eremin, and M. M. Korshunov, Phys. Rev. B **79**, 220501(R) (2009).

¹⁵ D. J. Scalapino and T. P. Devereaux, Phys. Rev. B **80**, 140512(R) (2009).

¹⁶ M. V. Klein, Phys. Rev. B **82**, 014507 (2010).

¹⁷ F. J. Burnell *et al.*, Phys. Rev. B **82**, 144506 (2010).